A Lattice Model for Emergent Supersymmetry in D=2 Topological Superconductors

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A class of topological superconductors in two dimensions support gapless Majorana edge modes that are protected by time-reversal symmetry. Here we study the quantum phase transition (QPT) associated with the destruction of these edge modes as magnetic order develops spontaneously along the edge. It was recently conjectured that this quantum critical point has space-time supersymmetry, that emerges dynamically by tuning only one parameter. Here, we put this assertion to test by constructing a one-dimensional lattice model of the edge, in which the role of time reversal is played by a non-local symmetry. We solve this model using state-of-the-art Density Matrix Renormalization Group (DMRG) method, and indeed find emergent supersymmetry at the symmetry breaking transition. In particular, we find that the QPT lies in the tricritical Ising universality, even though it is generically accessible by tuning only one parameter. We provide a detailed description of the QPT, and discuss the consequences of the emergent supersymmetry.

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Contents

I. INTRODUCTION

In this paper, we solve the problem of magnetic phase transition at the edge of a two dimensional time-reversal invariant topological superconductor (TSC)[1,2]. We accomplish this task by constructing a lattice model that captures the physics of this quantum phase transition (QPT)[3] and we solve the model using state-of-the-art Density Matrix Renormalization Group (DMRG)[4]. We find that the QPT has emergent space-time supersymmetry and lies in the tricritical Ising universality class[5–7], even though it is accessed by tuning only one parameter. This is in complete agreement with the recent field-theoretic arguments that addressed the same problem[8].

The discovery of $\mathbb{Z}_2$ topological insulators[9–15] has led to an explosion of activity in the field of topological phases. A complete classification of non-interacting topological phases[16,17] reveals a class of superconductors (class DIII TSC)[1,2,18,19], which supports Majorana edge states protected by time-reversal symmetry[16,17,20–27]. The well known B phase of superfluid He-3[28] is a realization of a topological phase in this class , and possible solid state realizations have also been proposed[29–31]. The Majorana edge modes of DIII TSC are the superconducting analogs of the celebrated Dirac fermion modes in topological insulators (TIs)[9–15]. Since these modes are symmetry-protected, spontaneous breaking of the time-reversal symmetry provides a natural route to gap them out, without closing the fermionic energy gap in the bulk. For example, electron correlations often lead to magnetic order, which breaks time reversal symmetry. A natural question is: how do the edge modes evolve as the magnetic order sets in?

In a recent work[8], it was argued that the QPT where the magnet order develops along the boundary of a DIII TSC has emergent space-time supersymmetry. The field theory for the QPT consisted of gapless bosonic modes of the critical magnetic fluctuations strongly coupled with the boundary Majorana edge states, such that the bosonic modes are superpartners of the Majorana fermions. Specifically, in d=2, it was argued that this is the tricritical Ising transition[5–7], *accessed by tuning a single parameter*. In this paper, we construct an explicit one-dimensional lattice model which, in the continuum limit, reproduces the theory of the boundary of a TSC coupled to the magnetic order parameter. We simulate this lattice model using state-of-the art density matrix renormalization group (DMRG) and find unambiguous verification of the claim in Ref.[8] that the QPT corresponds to (supersymmetric) Ising tricritical point and is accessed by tuning only one parameter. We note that other constructions of supersymmetric condensed matter models have appeared in the literature, but they all require tuning multiple parameters[32,33]. We emphasize that our situation is different, in that it exploits the special properties of the boundary states of topological superconductors, to access a supersymmetric state on tuning a single parameter.

A crucial property of our one-dimensional lattice model

is that, in the magnetically disordered phase, it supports gapless helical Majorana edge states of two-dimensional TSC. This in itself is an interesting feature because one might naively expect that the helical Majorana modes cannot be obtained in a strictly one-dimensional model without fine tuning. We achieve this task by observing that the free helical Majorana modes in one-dimension are dual to the critical point of a transverse field Ising model[34,35] which has an additional *self-duality symmetry*[35,36]. Our model retains this duality symmetry even after coupling to the magnetic degrees of freedom, which leads to protected helical Majorana modes without fine tuning, as long as the magnetic moments are in the disordered phase.

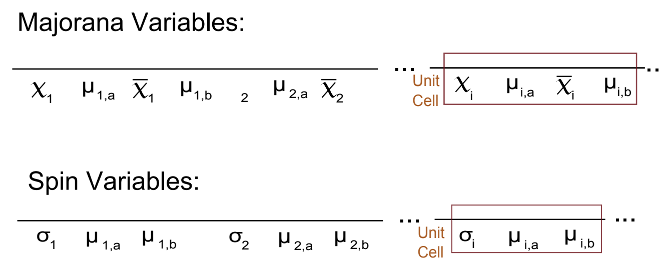

Figure 1: The unit cell of our lattice model consists of two Majorana fermions (χ and $\overline{\chi}$) and two spin-$\frac{1}{2}$ spins (μ_a and μ_b) or equivalently, three spin-$\frac{1}{2}$ spins (σ, μ_a and μ_b). The interactions between the spins are specified by the Hamiltonian H in the text (Eqn.5).

II. MAGNETIC QPT AT THE BOUNDARY OF TOPOLOGICAL SUPERCONDUCTOR

A. Field theory of the transition

In this section we introduce the problem of the magnetic instability at the edge of a two dimensional DIII-TSC[8]. Owing to the topological band structure of a TSC, there exist “helical Majorana modes” at its boundary[1,2]. This means that the edge states consist of non-interacting Majorana fermions χ_R, χ_L that move in the opposite directions and are described by the following quantum action:

$$S_\chi = \int d\tau dx \left[\chi_R \left(\partial_\tau + i\partial_x\right)\chi_R + \chi_L\left(\partial_\tau - i\partial_x\right)\chi_L\right] \quad (1)$$

where we have set the velocity of the Majorana fermions to unity. The above action has an inbuilt time-reversal symmetry $\mathcal{T}$ which acts on the Majorana fermions in the following way:

$$\chi_R \xrightarrow{\mathcal{T}} \chi_L, \chi_L \xrightarrow{\mathcal{T}} -\chi_R$$

. In quantum mechanics, the time-reversal operation is anti-unitary that is, it acts on the imaginary number $\sqrt{-1} \equiv i$ as $i \xrightarrow{\mathcal{T}} -i$. This implies that a potential mass term $H_{gap} = \int dx\, i\,\chi_L\chi_R$ is not allowed in the action since it changes sign under $\mathcal{T}$. This is the reason that the Majorana modes at the edge of a TSC stay gapless as long as the time-reversal symmetry is not broken.

We are interested in studying potential instabilities of the edge that gap out the Majorana modes. The magnetic fluctuations can be described via a conventional ϕ^4-theory for an Ising variable ϕ:

$$S_\phi = \int d\tau dx \, \left[\frac{1}{2}(\partial_\tau\phi)^2 + \frac{c^2}{2}(\vec{\nabla}\phi)^2 + \frac{r}{2}\phi^2 + \frac{u}{4}\phi^4\right] \quad (2)$$

Under time-reversal ϕ reverses its sign and therefore the coupling between the Majorana modes and the magnetic fluctuations, to the leading order, is given by:

$$S_{int} = 2ig\int d\tau dx \; \chi_L\chi_R\,\phi \quad (3)$$

S_{int} is clearly invariant under $\mathcal{T}$. The full action is given by

$$S = S_\chi + S_\phi + S_{int}. \quad (4)$$

Clearly, when Ising field ϕ orders, it spontaneously generates a mass gap $\propto g\langle\phi\rangle$ for the Majorana fermions. Our aim is to determine the critical properties of the corresponding QPT.

B. A lattice model for the transition

We now construct a novel one-dimensional lattice model that reproduces the physics of the magnetic phase transition described in the previous subsection. Naively, one might think that the edge of a two dimensional topological phase may not be realizable faithfully by a strictly one-dimensional model, and therefore one may need to simulate a full two-dimensional model to describe the QPT. We achieve this seemingly difficult task by exploiting the self-duality symmetry of the critical 1+1-d transverse field Ising model[35,36], which is also dual to the gapless helical Majorana modes via Jordan-Wigner transformation[34,35]. The self-duality symmetry simulates the aforementioned time-reversal symmetry $\mathcal{T}$ in a non-local manner.

The unit cell of our lattice model (Fig.1) consists of two Majorana fermions and two spin-$\frac{1}{2}$ spins: $\chi_i, \overline{\chi}_i, \vec{\mu}_{i,a}, \vec{\mu}_{i,b}$ or equivalently, three spin-$\frac{1}{2}$ spins : $\vec{\sigma}_i, \vec{\mu}_{i,a}, \vec{\mu}_{i,b}$ where the labels a and b serve to distinguish the two different $\vec{\mu}$ spins within the unit cell. Similar to the action S in Eqn.4, the Hamiltonian of our model consists of three terms:

$$H = H_1 + H_2 + H_3 \quad (5)$$

where

$$H_1 = -\sum_i \left(\sigma_i^z \sigma_{i+1}^z + \sigma_i^x\right) \quad (6)$$

$$= -i \sum_i \left(\chi_i \overline{\chi}_i + \overline{\chi}_i \chi_{i+1}\right) \quad (7)$$

$$H_2 = -\sum_i \left(\mu_{i,a}^z \mu_{i,b}^z + \mu_{i,b}^z \mu_{i+1,a}^z + h(\mu_{i,a}^x + \mu_{i,b}^x)\right) \quad (8)$$

$$H_3 = g \sum_i \left[\sigma_i^x (\mu_{i,a}^z + \mu_{i,b}^z) - \sigma_i^z \sigma_{i+1}^z (\mu_{i,b}^z + \mu_{i+1,a}^z)\right] \quad (9)$$

$$= ig \sum_i \left(\chi_i \overline{\chi}_i (\mu_{i,a}^z + \mu_{i,b}^z) - \overline{\chi}_i \chi_{i+1} (\mu_{i,b}^z + \mu_{i+1,a}^z)\right)$$

Above, the Eqns.7 and 10 follow from Eqns.6 and 9 via the well-known Jordan Wigner transformation on the spin variables[34,35].

One can also write down a slightly simpler model where H_3 is replaced by

$$H_3' = ig \sum_i \left(\chi_i \overline{\chi}_i \mu_{i,a}^z - \overline{\chi}_i \chi_{i+1} \mu_{i,b}^z\right) \quad (10)$$

$$= g \sum_i \left(\sigma_i^x \mu_{i,a}^z - \sigma_i^z \sigma_{i+1}^z \mu_{i,b}^z\right) \quad (11)$$

Above, H_1 is the transverse field Ising model *at criticality* for the σ spins which is dual to the free Majorana fermions (Eqn.7) via Jordan-Wigner transformation[34,35]. Therefore, its continuum limit corresponds to the helical Majorana mode action S_χ in Eqn.1. H_2 corresponds to the transverse field Ising model for the μ spins at a transverse field h. In the continuum limit, the corresponding action is given by S_ϕ (Eqn.2). Finally, H_3 describes the coupling between the two Ising models and precisely reproduces the term S_{int} in the continuum limit. The parameter h is the analog of the r term in action S in Eqn.4 while we retain the notation g for the coupling between the two Ising models as before.

One might naively expect that in our model, the $\vec{\sigma}$ spins would generically be off-critical since after coupling to the $\vec{\mu}$ spins, the coefficient of $-\sum_i \sigma_i^z \sigma_{i+1}^z$ term in H_1 may renormalize and become different than that of the term $-\sum_i \sigma_i^x$. This will spoil the correspondence with the helical Majorana modes which stay gapless as long as scalar field ϕ does not condense. Interestingly, the above models (with either H_3 or H_3' term) have an intricate symmetry, which we denote by $\mathcal{T}'$, that ensures that $\vec{\sigma}$ spins stay gapless throughout the whole phase when $\vec{\mu}$ spins are disordered. In particular, the above model is invariant under: $\sigma_i^z \sigma_{i+1}^z \xrightarrow{\mathcal{T}'} \sigma_{i+1}^x$, $\sigma_i^x \xrightarrow{\mathcal{T}'} \sigma_i^z \sigma_{i+1}^z$, $\mu_{i,a}^z \xrightarrow{\mathcal{T}'} -\mu_{i,b}^z$, $\mu_{i,b}^z \xrightarrow{\mathcal{T}'} -\mu_{i+1,a}^z$, $\mu_{i,a}^x \xrightarrow{\mathcal{T}'} \mu_{i,b}^x$, $\mu_{i,b}^x \xrightarrow{\mathcal{T}'} \mu_{i+1,a}^x$. Physically, this symmetry exploits the fact that the transverse field Ising model is self-dual at the criticality. The self-duality interchanges the terms $-\sum_i \sigma_i^z \sigma_{i+1}^z$ and $-\sum_i \sigma_i^x$ in H_1 and this is precisely the action of the symmetry $\mathcal{T}'$ on the $\vec{\sigma}$ spins. Since the symmetry is respected by the full interacting Hamiltonian $H = H_1 + H_2 + H_3$ (or $H' = H_1 + H_2 + H_3'$), the spins $\vec{\sigma}$ remain gapless unless the $\mathcal{T}'$ symmetry is broken spontaneously. This happens when $\langle \mu_z \rangle \neq 0$, in the exact analogy with the action S in Eqn.4 where $\langle \phi \rangle$ provides mass gap to the helical Majorana modes. Thus, for a given coupling g, as one varies h from large to small values, one begins in the phase where the μ spins are disordered, implying a gapless phase with $c = 1/2$. On lowering h, there is an ordering transition at $h_c(g)$, below which the μ spins order, i.e. $\langle \mu_z \rangle \neq 0$. This leads to the gapped, symmetry broken phase. The critical point at $h_c(g)$, as in Figure 4, is the primary focus of this work. We note that the unusual symmetry of the model under study is reflected in the fact that the disordered state of the spin model has a three fold degenerate ground state, since one of the two degenerate ordered states of μ^z corresponds to the ordered phase of the σ^z, which is itself two fold degenerate[48]. Thus, in this respect it resembles the first order line between the disordered and and ordered phases in Ref[5].

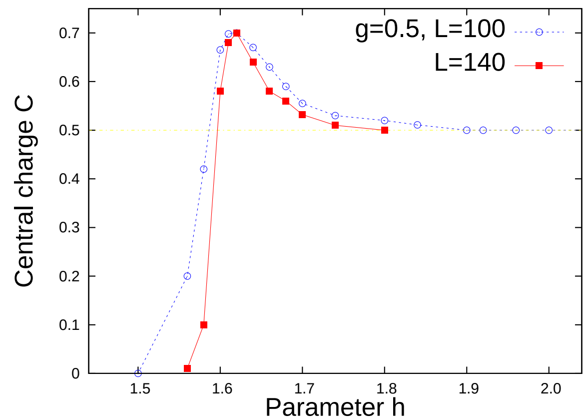

Figure 2: The evolution of the central charge for $g = 0.5$ as the parameter h is varied in Hamiltonian H (Eqn.5). The central charge jumps from $c = 0$ to $c = 7/10$ as one approaches the transition between the two phases (the phase diagram is shown in Fig.4). It settles down to $c = 1/2$ on the other side of the transition since the corresponding phase is described by the critical point of the 1+1-d transverse field Ising model. We note that the jump from $c = 0$ to $c = 7/10$ becomes sharper as the total system size increases, since the finite size effects decrease.

Having argued that our model reproduces the physics of the action S (Eqn.4), universality of the critical phenomena implies that it can also be used to study the critical properties of the magnetic phase transition, to which we now turn.

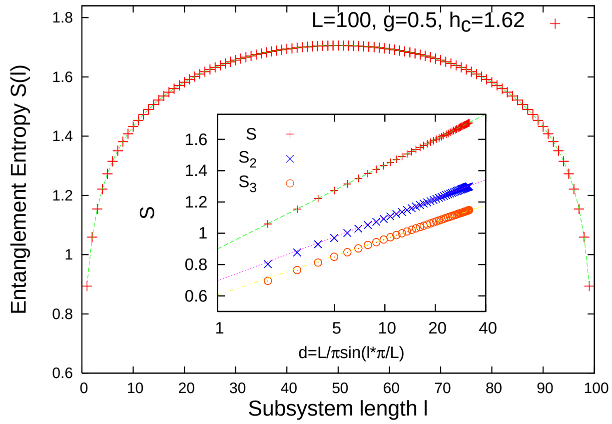

Figure 3: The scaling of the von-Neumann entanglement entropy and the Renyi entropies S_2 and S_3 at the critical point between the two phases in Fig.4. The scaling of the entanglement entropy fits well to the predictions from analytical calculations (Eqn.12) with a central charge $c = 7/10$, confirming that the transition lies in the tricritical Ising universality.

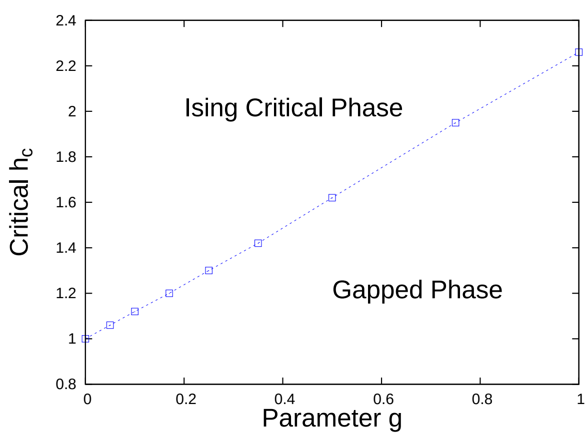

Figure 4: The phase diagram of the model in Eqn.5 obtained using the DMRG method. The Ising critical phase is identified by a gapless CFT with central charge $c = 1/2$ and disorderd $\vec{\mu}$ spins, while the gapped phase corresponds to ordering of $\vec{\mu}$ spins which interact with the $\vec{\sigma}$ spins via the term H_3 in Eqn.5 to gap them out. The line separating the two phases correspond to tricritical Ising CFT with central charge $c = 7/10$. The central charge along the whole $g = 0$ line for $h \neq 1$ is $1/2$ since the $\vec{\sigma}$ and $\vec{\mu}$ spins decouple along this line, while the point $g = 0, h = 1$ corresponds to a multicritical point with central charge unity since the μ spins are also at the Ising critical point.

III. NUMERICAL SOLUTION VIA DMRG AND THE EMERGENT SUPERSYMMETRY

We solve the Hamiltonian H in Eqn.5 numerically using DMRG[4] calculations. We simulate lattice sizes upto 140 unit cells with periodic boundary conditions[49] and keep up to 1024 states to ensure the convergence of the entanglement entropy in DMRG optimization. For smaller system with 8 unit cells, we have checked that our DMRG results are identical to exact diagonalization results. The DMRG is a very useful tool for our purposes since it provides a direct access to the *central charge* of a theory, which uniquely determines the critical theory in our model. To this end, we note that one of the primary outputs from a DMRG calculation is the von-Neumann entanglement entropy $S(l)$ for a bipartition of the system of size L into two subsystems of sizes l and $L - l$. For a subsystem A, it is defined as $S(A) = -\text{Tr } \rho_A \log \rho_A$ where ρ_A is the reduced density matrix for the subsystem A obtained by tracing out the rest of the system. For a conformal field theory (CFT), $S(l)$ is known to scale as[37]:

$$S(l) = \frac{c}{3} \log \left(\frac{L}{\pi} \sin \left(\frac{\pi l}{L} \right) \right) + \mathrm{O}(1) \qquad (12)$$

where c is the central charge of the CFT and $\mathrm{O}(1)$ denotes subleading corrections. One observes that the above expression is symmetric under $l \leftrightarrow L - l$ at the leading order which is a consequence of the fact that $S(A) = S(\overline{A})$ for an arbitrary subregion A. For a gapped phase, $c = 0$ and the entanglement entropy saturates to a constant value as l increases. One may also look at higher moments of the reduced density matrix, which go by the name of "Renyi entropies" $S_n = -\frac{1}{n-1} \log \text{Tr } \rho_A^n$. For a CFT, $S_n(l)$ scales similar to $S(l)$ in Eqn.12[37],

$$S_n(l) = \frac{c}{3} \left(1 + \frac{1}{n} \right) \log \left(\frac{L}{\pi} \sin \left(\frac{\pi l}{L} \right) \right) + \mathrm{O}(1) \qquad (13)$$

One may easily show that all the central charges involved in the problem of our interest (Eqn.5) are less than unity[8], and this uniquely determines the corresponding CFT[38,39].

Fig.2 shows the evolution of the central charge for $g = 0.5$ as the parameter h is varied. One observes that the central charge stays at zero at small h indicating a gapped phase. It jumps to ≈ 0.7 at $h_c \approx 1.62$ and then settles down to ≈ 0.5 at $h > 1.62$ side. We further note that the jump from $c = 0$ to $c = 7/10$ becomes sharper as the system size increases from $L = 100$ to 140 with the decrease of the finite size effect. Fig.3 shows the entanglement entropy as a function of the subsystem size l at the critical point between the two phases, that is, at $g = 0.5$ and $h_c = 1.62$. We find that von-Neumann entropy $S(l)$ fits perfectly to Eqn.12 to yield a central charge $c = \frac{7}{10}$ at the critical point. Furthermore using $c = \frac{7}{10}$ by fitting $S(l)$, we demonstrate the agreement

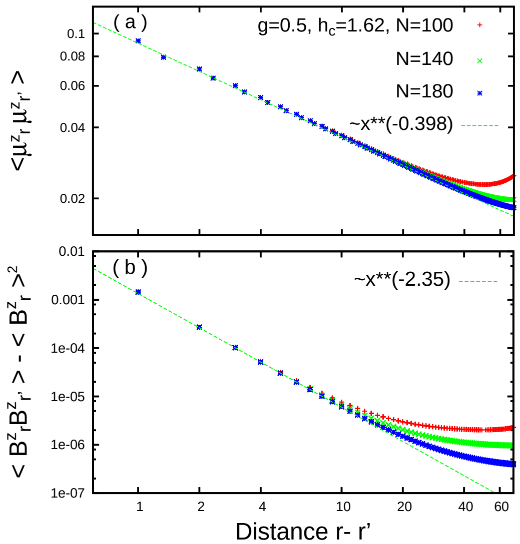

Figure 5: The top figure shows the scaling of the correlation function $\langle \mu^z_r \mu^z_{r'} \rangle$ while the bottom one shows the bond-bond correlation function $\langle \mu^z_r \mu^z_{r+1} \mu^z_{r'} \mu^z_{r'+1} \rangle - \langle \mu^z_r \mu^z_{r+1} \rangle^2$ (we denote the bond operator $B^z_r = \mu^z_r \mu^z_{r+1}$ in the label of the figure) for the Hamiltonian H (Eqn.5) at the critical point between the two phases in Fig.4. Emergent supersymmetry at the critical point implies that the difference in the power-law exponents for these two correlation functions is exactly two while the precise values of the exponents themselves are also found to be consistent with the tricritical Ising model (see the main text for details).

between numerical results and CFT predicted Renyi entropy S_n in the inset of the Fig.3. The central charge $c = \frac{7}{10}$ between the Ising critical phase and the gapped phase implies that the critical point associated with the magnetic ordering corresponds to tricritical Ising theory, in agreement with the field theoretic arguments in Ref.[8].

By repeating the same procedure as above for different values of the parameter g, one can obtain the whole phase diagram in the $g - h$ plane (Fig.4). Consistent with our prior assertions, our lattice model exhibits a whole phase at larger h side that is Ising critical without any fine-tuning and thus faithfully captures the edge theory of a two-dimensional TSC. This phase is separated by a line in the $g - h$ plane that describes the phase transition between the Ising critical phase and a fully gapped phase that breaks the $\mathcal{T}'$ symmetry spontaneously due to the condensation of μ_z spins. Most importantly, we verify the scaling of entanglement entropy, Eqn.12 with $c = \frac{7}{10}$, along the whole critical line that separates the two phases. *This provides an unambiguous evidence that the critical theory is given by the tricritical Ising model.*

Emergent Supersymmetry and its Consequences:

The tricritical Ising model in two dimensions is rather special: it has *space-time supersymmetry*[7,40,41]. Indeed, the form of the action S in Eqn.4 is ready-made for $\mathcal{N} = 1$ supersymmetry in 1+1-d, which is the theory of a single real fermion coupled to a single real scalar via Yukawa interaction[42] (Eqn.3) (the reader may note that in supersymmetric notation, the doublet of fermions $\chi = [\chi_R \ \chi_L]^T$ counts as a single fermion). Furthermore, for massless fermions and bosons, the $\mathcal{N} = 1$ supersymmetric Wess-Zumino model is known to be equivalent to the tricritical Ising model. To see how supersymmetry may possibly be realized by the action S in Eqn.4, let us set the bare value of the couplings $r = 0, c = 1$ and further assume that the relation $u = \frac{g^2}{2}$ holds . If so, one may easily verify that S is invariant under the following 'fermionic rotation'[42]:

$$\delta\phi = \tilde{\epsilon}\chi \tag{14}$$

$$\delta\chi = (\frac{-i\gamma_\mu \partial_\mu \phi}{2} + i\frac{g}{4}\phi^2)\epsilon \tag{15}$$

Here $\epsilon = [\epsilon_1 \ \epsilon_2]^T$ is an arbitrary two-component Majorana variable, $\tilde{\epsilon} = \epsilon\, i\sigma_y$, $\chi = [\chi_R \ \chi_L]$, $\gamma_\tau = \sigma_y$, $\gamma_x = \sigma_x$, and the Pauli matrices $\vec{\sigma}$ act on the spinor index of fermions χ and ϵ. The numerical solution of our lattice model demonstrates that $\mathcal{N} = 1$ supersymmetry emerges dynamically at the QPT by tuning only one parameter (in contrast to the fine tuning of four parameters: $r, c, \frac{g^2}{u}$ and the Majorana fermion mass). Supersymmetry implies that the velocity as well as the anomalous dimension of the Majorana fermion χ equals that of the scalar ϕ, a direct consequence of invariance under 'fermionic rotation'[42], Eqns.14 and 15. In particular,

$$\langle \chi_L(r)\chi_L(0) \rangle \sim \langle \chi_R(r)\chi_R(0) \rangle \sim \frac{1}{r^{1+\eta_1}} \tag{16}$$

$$\langle \phi(r)\phi(0) \rangle \sim \frac{1}{r^{\eta_2}} \tag{17}$$

with $\eta_1 = \eta_2 = 0.4$. Supersymmetry also implies that the difference of the scaling dimension of the operators μ^z_i and $\mu^z_i \mu^z_{i+1}$ is precisely unity[39]. Specifically[39], $\langle \mu^z_r \mu^z_{r'} \rangle \sim \frac{1}{|r-r'|^{0.4}}$ while $\langle \mu^z_r \mu^z_{r+1} \mu^z_{r'} \mu^z_{r'+1} \rangle - \langle \mu^z_r \mu^z_{r+1} \rangle^2 \sim \frac{1}{|r-r'|^{2.4}}$. We verified this particular prediction in our DMRG simulations and the results are shown in Fig.5.

IV. CONCLUSIONS

In this paper, we studied the quantum phase transition (QPT) associated with the magnetic instability of the boundary states in two dimensional DIII class topological superconductors (TSC). We found that this single-parameter tuned transition has emergent space-time supersymmetry at the critical point and lies in the tricritical Ising universality class, in agreement with the recent field-theoretic arguments. To derive this assertion, we constructed a concrete lattice model for the phase transition which we solved using the DMRG method. A physical scenario leading to such a transition can arise in several ways. One can imagine a situation where the

material has two bands, one of them contributing itinerant electrons that form a topological superconductor while the other band consists of localized moments that provide the Ising magnetic variable. Alternatively, one can also dope the system with magnetic atoms in a uniform manner so that the disorder effects can be ignored. In the latter scenario, preferentially doping the surface could trigger a surface transition while the bulk retains time reversal symmetry.

We note that from a field theoretic point of view a supersymmetric fixed point is generically multicritical. For example, the action S is supersymmetric when $r = 0, u = \frac{g^2}{2}$, $c = 1$ and the Majorana fermion mass is also set to zero. Indeed, there already exist lattice models in literature that support supersymmetric multicritical points[32,33]. What is special about our model is that it leads to supersymmetry by tuning only one parameter due to its intrinsic dynamics. We note that higher dimensional analogs of similar phenomena have also been proposed[8,43–47] and it would be interesting to realize lattice model in higher dimensions that can be numerically simulated and exhibit the phenomena of emergent supersymmetry.

The emergent supersymmetry at the criticality has many remarkable consequences. We already mentioned that it leads to the equality of the anomalous scaling dimension of the Majorana fermion (which can be extracted using ARPES experiments) and the anomalous dimension of the magnetic order parameter (measured using neutron scattering experiments). It also implies emergent Lorentz invariance at the criticality, which leads to the equality of the speed of the Majorana fermion and the critical fluctuations associated with the magnetic order parameter. Finally, we note that one can also imagine situations where the magnetic order sets throughout the bulk of the TSC; and we refer the reader to a separate analysis of such transitions[8].

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1 R. Roy, arXiv:0803.2868.
2 X.-L. Qi, T. L. Hughes, S. Raghu, and S.-C. Zhang, Phys. Rev. Lett. 102, 187001 (2009).
3 S. Sachdev, Quantum Phase Transitions, Cambridge University Press, 2^{nd} Ed.,(2011).
4 S. R. White, Phys. Rev. Lett. 69, 2863 (1992).
5 D. A. Huse, Phys. Rev. Lett. 49, 1121 (1982); Phys. Rev. B 30, 3908 (1984).
6 A. A. Belavin, A. B. Zamolodchikov, and A. M. Polyakov, J. Stat. Phys. 34, 763 (1984).
7 D. Friedan, Z. Qiu, S. Shenker, Phys. Rev. Lett. 52, 1575 (1984).
8 T. Grover, A. Vishwanath, arXiv:1206.1332.
9 M. Konig et al. Science 318, 766 (2007).
10 D. Hsieh et al., Nature (London) 452, 970 (2008).
11 D. Hsieh et al., Science 323, 919 (2009).
12 Y. Xia et al. Nat. Phys. 5, 398 (2009).
13 Y. L. Chen et al., Science 325, 178 (2009).
14 P. Roushan et al., Nature (London) 460, 1106 (2009).
15 D. Hsieh et al., Nature (London) 460, 1101 (2009).
16 Andreas P. Schnyder, Shinsei Ryu, Akira Furusaki, Andreas W. W. Ludwig, Phys. Rev. B 78, 195125 (2008);
17 A. Kitaev, Proceedings of the L.D.Landau Memorial Conference “Advances in Theoretical Physics” (2008).
18 G. E. Volovik, *The Universe in a Helium Droplet*, Oxford University Press (2003).
19 G. E. Volovik, *Exotic Properties of Superfluid He-3*, World Scientific (1992).
20 M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. 82, 3045 (2010); X. L. Qi and S. C. Zhang, Rev. Mod. Phys. 83, 1057 (2011). M. Z. Hasan and J. E. Moore, Annual Review of Condensed Matter Physics Vol. 2, 55 (2011).
21 C. L. Kane and E. J. Mele, Phys. Rev. Lett. 95, 146802 (2005).
22 B. A. Bernevig, T. L. Hughes, and S.-C. Zhang, Science 314, 1757 (2006).
23 L. Fu, C. L. Kane, E. J. Mele, Phys. Rev. Lett. 98, 106803 (2007).
24 J. E. Moore, L. Balents, Phys. Rev. B 75, 121306(R) (2007).
25 R. Roy, Phys. Rev. B 79, 195322 (2009).
26 X.-L. Qi, T. L. Hughes, and S.-C. Zhang, Phys. Rev. B 78, 195424 (2008).
27 Shinsei Ryu, Andreas Schnyder, Akira Furusaki, Andreas Ludwig, New J. Phys. 12, 065010 (2010).
28 A. Leggett, Rev. Mod. Phys. 47, 331 (1975).
29 L. Fu, E. Berg, Phys. Rev. Lett. 105, 097001 (2010).
30 Yuan-Yuan Xiang, Wan-Sheng Wang, Qiang-Hua Wang, Dung-Hai Lee, arXiv:1201.2003.
31 S. Raghu, A. Vishwanath (manuscript in preparation).
32 Paul Fendley, Kareljan Schoutens, Jan de Boer, Phys.Rev.Lett. 90, 120402 (2003).
33 Bela Bauer, Liza Huijse, Erez Berg, Matthias Troyer, Kareljan Schoutens, arXiv:1208.0343.
34 E Lieb, T Schultz, D Mattis, Rev. Mod. Phys. 36, 856 (1964).
35 J. B. Kogut, Rev. Mod. Phys. 51, 659 (1979).
36 H. A. Kramers and G. H. Wannier, Phys. Rev. 60, 252 (1941).
37 P. Calabrese, J. Cardy, J. Stat. Mech. , P06002 (2004). C. Callan, F. Wilczek, Phys. Lett. B 333, 55 (1994). Holzhey, C., F. Larsen, and F. Wilczek, Nucl. Phys. B 424, 443 (1994).
38 A. Zamolodchikov, JETP Lett 43, pp 730-732 (1986).
39 P. D. Francesco, P. Mathieu, D. Senechal, *Conformal Field Theory*, Springer 1996.
40 A. Zamolodchikov, Sov. J. Nucl. Phys. 44, 529 (1986).
41 A. Zamolodchikov, Nucl. Phys. B 358, 524 (1991).
42 J. Wess and J. Bagger, *Supersymmetry and Supergravity*, Princeton University Press (1992).

[43] L. Balents, M.P.A. Fisher and C. Nayak, Intnl. J. Mod. Phys. B 12, 1033 (1998).
[44] Sung-Sik Lee, Phys. Rev. B 76, 075103 (2007).
[45] Hidenori Sonoda, Prog. Theor. Phys. 126, 57 (2011).
[46] Scott Thomas' KITP talk (2005), `http://online.itp.ucsb.edu/online/qpt_c05/thomas/`
[47] P. Ponte, S.-S. Lee, arXiv:1206.2340.
[48] We thank David Huse for discussion on this point.
[49] Strictly, the duality from fermions to the Ising chain dictates specific boundary conditions, which however are unimportant for the physical quantities we study here. Hence we always work with periodic boundary conditions.